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WAVE MOTIONS IN A SPATIAL BOUNDARY LAYER†

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The propagation of perturbations in a boundary layer under conditions when the velocity of the approaching stream may be both subsonic and supersonic is considered. With regard to the initial flow in the boundary layer it is assumed that it is stationary and possesses a spatial character which is caused by the external pressure gradient and not by the curvature of the body around which the flow occurs (boundary layers of this kind are extensively used in experiments at the present time). The linearized equations describing waves of vanishingly small amplitude are studied in detail. An analysis of the dispersion relation which links the frequency of the free oscillations with the components of the wave vector reveals a number of special features which are only present in motions with a three-dimensional velocity field. In particular, it is established that the Cauchy problem for the system of linear equations is ill-posed.

AS IS WELL known, the propagation of perturbations of any geometric configurations in an initially stationary two-dimensional flow obeys the equations of an interacting boundary layer in which the self-induced pressure gradient is subject to determination together with the velocity field. The asymptotic equations of the theory of free interaction have been derived in connection with an analysis of the motion of a fluid in the neighbourhood of the rear edge of a plate of finite length [1, 2] and of the detachment of a supersonic boundary layer from the surface around which the flow occurs [3, 4]. Later, spatial perturbations were considered both under stationary conditions [5] and as it applies to stability problems [6]. The latter area has been under extensive development and a review of the corresponding results is given in [7].

The asymptotic analysis in the papers mentioned above is based on the assumption that the Reynolds number $R \to \infty$, whence it follows that it is possible to introduce a small parameter $\varepsilon = R^{1/2}$. The Reynolds number is calculated using the distance L^* from the leading edge of the plate, the velocity $U_\infty{}^*$, the density $\rho_\infty{}^*$ and the first coefficient of viscosity $\lambda_{1\infty}{}^*$ of the particles in the approach stream which are subjected to compression by a pressure $P_\infty{}^*$. If the initial boundary layer possesses a spatial structure, it is necessary to interpret L^* as being a certain characteristic dimension and $U_\infty{}^*$, $\rho_\infty{}^*$, $P_\infty{}^*$ and $\lambda_\infty{}^*$ as being the corresponding external flow parameters at the point being considered on the surface around which the flow occurs. With such an approach, the Reynolds number R acquires a local character and can vary depending on the position of the chosen point. Changes in the velocity field and the thermodynamic functions in the external non-viscous flow occur, as a rule, due to warping of the surface around which the flow occurs.

On the other hand, in recent times models in which the surface being studied with a boundary layer adjacent to it is planar but a pressure gradient is created artificially using various devices [8, 9] are being ever more frequently employed in experiments. Such a technique has the advantage that it

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enables one to avoid the effect of centrifugal forces, acting close to warped segments of the surface, on the propagation of perturbations within the boundary layer. A generalization of the theory developed in [5, 6], which includes spatial boundary layers of the above-mentioned type, has been outlined in [10, 11] and is developed below with the aim of elucidating new qualitative effects.

1. THE EQUATIONS OF WAVE MOTIONS

As is customary, we will subdivide the whole field of flow into three characteristic domains which are arranged one above another. In the upper domain, the effects of viscosity and thermal conductivity are not very pronounced and the perturbations of all the gas parameters will therefore be quantities of the same order of smallness. The central domain, which occupies almost the whole of the thickness of the boundary layer, to a first approximation, is also not subject to the influence of dissipative factors but it is substantially swirled and the lateral component of the velocity is comparable with its component in the direction of the main (external) flow. As far as the lower domain which is close to the wall is concerned, viscosity plays a decisive role in the formation of its structure while allowance for the thermal conductivity only becomes important when there is non-uniform heating of the surface.

We denote the time by t^*, place the origin of the Cartesian system of coordinates x^*, y^* and z^* in a plane with the x^* axis directed along the velocity vector of the external unperturbed flow at the point considered and line up the y^* axis with a normal to the plane. As projections onto the axes of the selected system of coordinate, the velocity components will be u^*, v^* and w^*. Furthermore, let ρ^* be the density and P^* be the pressure of the gas. Since our main interest lies in the study of the perturbations which propagate in a spatial boundary layer, we shall commence the analysis with the central layer, which includes the greater part of its thickness. Here, let us put

$$t^* = \varepsilon^2 (L^*/U_\infty^*)\, t', \quad x^* = \varepsilon^3 L^* x', \quad y^* = \varepsilon^4 L^* y_2, \quad z^* = \varepsilon^3 L^* z' \tag{1.1}$$

and normalize the required functions in the following manner:

$$\begin{aligned} u^* &= U_\infty^* \,[U_{x0}(y_2) + \varepsilon u_2 (t', x', y_2, z') + \ldots] \\ v^* &= U_\infty^* \,[\varepsilon^2 v_2 (t', x', y_2, z') + \ldots] \\ w^* &= U_\infty^* \,[U_{z0}(y_2) + \varepsilon w_2 (t', x', y_2, z') + \ldots] \\ \rho^* &= \rho_\infty^* \,[R_0(y_2) + \varepsilon \rho_2 (t', x', y_2, z') + \ldots] \\ P^* &= P_\infty^* + \rho_\infty^* U_\infty^{*2} \,[\varepsilon^2 p_2 (t', x', y_2, z') + \ldots] \end{aligned} \tag{1.2}$$

In expansions (1.2) the quantities U_{x0}, U_{z0} and R_0 are assumed to be known from a solution of the global problem of calculating a stationary boundary layer on a plane surface, the pressure gradient around which is created by any supplementary devices. It is obvious that U_{z0} is the velocity of the so-called secondary flow and, by virtue of the choice of the directions of the coordinate axes $U_{x0} \to 1$, $U_{z0} \to 0$ and $y_2 \to \infty$.

By taking account of the scales of time and the spatial variables, which are defined by (1.1), and introducing the expansions (1.2) into the Navier–Stokes system of equations, we have

$$\frac{\partial u_2}{\partial x'} + \frac{\partial v_2}{\partial y_2} + \frac{\partial w_2}{\partial z'} = 0$$

$$U_{x0} \frac{\partial u_2}{\partial x'} + v_2 \frac{\partial U_{x0}}{d y_2} + U_{z0} \frac{\partial u_2}{\partial z'} = 0$$

$$U_{x0}\frac{\partial w_2}{\partial x'}+v^2\frac{dU_{z0}}{dy_2}+U_{z0}\frac{\partial w_2}{\partial z'}=0$$
$$\frac{\partial p_2}{\partial y_2}=0,\quad U_{x0}\frac{\partial \rho_2}{\partial x'}+v_2\frac{dR_0}{dy_2}+U_{z0}\frac{\partial \rho_2}{\partial z'}=0 \tag{1.3}$$

Here, the first three equations are separated from the remaining equations, which enables us to find the velocity field of the perturbations regardless of the changes in the excess density. We will now introduce an instantaneous displacement thickness $A'(t', x', z')$ into the treatment. In terms of this thickness the result of integration of the system of equations (1.3) reads

$$u_2=A'\frac{dU_{x0}}{dy_2},\quad w_2=A'\frac{dU_{z0}}{dy_2},\quad \rho_2=A'\frac{dR_0}{dy_2}$$
$$v_2=-\frac{\partial A'}{\partial x'}U_{x0}-\frac{\partial A'}{\partial z'}U_{z0} \tag{1.4}$$

The difference between the solution which has been constructed and that which determines the propagation of perturbations of an arbitrary configuration in the initial two-dimensional boundary layer is contained in the expression for the lateral component of the velocity of the particles. If $U_{z0}=0$, this component becomes of the order of ε^2 and occurs, as follows from [5, 6], because $\partial p_2/\partial z' \neq 0$. The relationship between w_2 and p_2 in the case of waves travelling through a spatial boundary layer is established by the law governing their interaction with the external flow since an excess pressure is induced when there is a change in the displacement thickness.

In order to formulate the above-mentioned law, let us consider the exterior flow domain. In this domain, the normalization of the time and the coordinate in the plane around which the flow occurs remains as before and the distance to it from a selected point in space is introduced by the relationship $y^* = \varepsilon^3 L^* y_1$. The perturbations of all the gas parameters are proportional to ε^3, whence

$$u^* = U_\infty^*[1+\varepsilon^2 u_1(t', x', y_1, z')+\ldots],\quad v^* = U_\infty^*[\varepsilon^2 v_1(t', x', y_1,$$
$$z')+\ldots],\quad w^* = U_\infty^*[\varepsilon^2 w_1(t', x', y_1, z')+\ldots],\quad \rho^* = \rho_\infty^*[1+\varepsilon^2\rho \tag{1.5}$$
$$(t', x', y_1, z')+\ldots],\quad P^* = P_\infty^* + \rho_\infty^* U_\infty^{*2}[\varepsilon^2 p_1(t', x', y_1, z')+\ldots]$$

As a result of substituting the expansions which have been written out into the system of Navier–Stokes equations and retaining just the leading terms in them, the problem reduces to the integration of a single equation for the excess pressure

$$(1-M_\infty^2)\frac{\partial^2 p_1}{\partial x'^2}+\frac{\partial^2 p_1}{\partial y_1^2}+\frac{\partial^2 p_1}{\partial z'^2}=0 \tag{1.6}$$

after which the components of the velocity vector of the particles and the variation in the density are calculated using the formulas

$$u_1=-p_1,\quad \rho_1=M_\infty^2 p_1$$
$$v_1=-\int_{-\infty}^{x'}\frac{\partial p_1(t',\xi,y_1,z')}{\partial y_1}d\xi,\quad w_1=-\int_{-\infty}^{x'}\frac{\partial p_1(t',\xi,y_1,z')}{\partial z'}d\xi \tag{1.7}$$

It is assumed that the perturbations decay upwards through the flow, that is, as $x' \to -\infty$. The Mach number is denoted by M_∞. Let us specify the vertical component of the velocity $v_1(t', x', 0, z')$ in the plane $y_1 = 0$. The result of integrating Eq. (1.6), taking account of the relation between p_1 and v_1, which is introduced by the second of the relationships (1.7), takes the form

$$p_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{\partial v_1(t', \xi, 0, \zeta)/\partial \xi}{\{(x'-\xi)^2 + (1 - M_\infty^2)[y_1^2 + (z'-\zeta)^2]\}^{1/2}} d\zeta \tag{1.8}$$

subject to the condition that $M_\infty < 1$. If the Mach number $M_\infty > 1$, then

$$p_1 = \frac{1}{\pi} \int_{-\infty}^{x'-\beta y_1} d\xi \int_{-\delta'+z'}^{\delta'+z'} \frac{\partial v_1(t', \xi, 0, \zeta)/\partial \xi}{\{(x'-\xi)^2 - \delta^2[y_1^2 + (z'-\zeta)^2]\}^{1/2}} d\zeta$$

$$\beta = \sqrt{M_\infty^2 - 1}, \quad \delta' = \delta'(x', \xi, y_1; M_\infty) = \left[\left(\frac{x'-\xi}{\beta}\right)^2 - y_1^2\right]^{1/2} \tag{1.9}$$

The expansions (1.2) for the main thickness of the boundary layer and the expansions (1.5) for the potential flow domain join together when $y_2 \to \infty$ and $y_1 \to 0$. On introducing the limiting values of U_{x0} and U_{z0} into the last of formulas (1.4), we obtain

$$v_1(t', x', 0, z') = -\frac{\partial A'}{\partial x'}, \quad p_1(t', x', 0, z') = p_2(t' x', z') \tag{1.10}$$

which repeats the results which refer to spatial perturbations which propagate in a compressed Blasius boundary level [5, 6].

It remains to consider the lower domain in which $y^* = \varepsilon^2 L^* y_3$. Here, the required functions are expanded in standard sequences.

$$u^* = U_\infty^* [\varepsilon u_3(t', x', y_3, z') + \ldots], \quad v^* = U_\infty^* [\varepsilon^3 v_3(t', x', y_3, z') + \ldots], \quad w^* = U_\infty^* [\varepsilon w_3(t', x', y_3, z') + \ldots], \quad \rho^* = \rho_\infty^* [\rho_3(t', x', y_3, z') + \ldots], \quad P^* = P_\infty^* + \rho_\infty^* U_\infty^{*2} [\varepsilon^2 p_3(t', x', y_3, z') + \ldots] \tag{1.11}$$

the nature of which can be simplified from the very outset by confining oneself to an analysis of purely mechanical oscillations and excluding internal temperature waves from it. As follows from [1–4], in this case $\rho_3 = R_0(0)$ and the coefficient of viscosity $\lambda_1^* = \lambda_{1w}^* = \lambda_1^*[R_0(0)]$, where the subscript w refers to the value of this coefficient in the plane around which the flow occurs. For simplicity, let us assume that it is thermally insulated and that Chapman's law $\lambda_{1w}^*/\lambda_{1\infty}^* = CT_w^*/T_\infty^*$, which relates the coefficient of viscosity to the ratio of the wall temperature T_w^* and the temperature T_∞^* of the surrounding space, holds. It is clear that $T_w^*/T_\infty^* = R_0^{-1}(0)$. When account is taken of the remarks which have been made, substitution of expansions (1.1) into the initial system of Navier–Stokes equations leads to the relationships

$$\frac{\partial u_3}{\partial x'} + \frac{\partial v_3}{\partial y_3} + \frac{\partial w_3}{\partial z'} = 0, \quad \frac{\partial p_3}{\partial y_3} = 0$$

$$R_0(0)\left(\frac{\partial u_3}{\partial t'} + u_3 \frac{\partial u_3}{\partial x'} + v_3 \frac{\partial u_3}{\partial y_3} + w_3 \frac{\partial u_3}{\partial z'}\right) = -\frac{\partial p_3}{\partial x'} + \frac{C}{R_0(0)} \frac{\partial^2 u_3}{\partial y_3^2} \tag{1.12}$$

$$R_0(0)\left(\frac{\partial w_3}{\partial t'} + u_3 \frac{\partial w_3}{\partial x'} + v_3 \frac{\partial w_3}{\partial y_3} + w_3 \frac{\partial w_3}{\partial z'}\right) = -\frac{\partial p_3}{\partial z'} + \frac{C}{R_0(0)} \frac{\partial^2 w_3}{\partial y_3^2}$$

Joining of the expansions (1.2) and (1.11) gives the limiting conditions

$$
\begin{aligned}
u_3 - \frac{\partial U_{x0}(0)}{dy_2} y_3 &\to \frac{dU_{x0}(0)}{dy_2} A'(t', x', z') \\
w_3 - \frac{dU_{z0}(0)}{dy_2} y_3 &\to \frac{dU_{z0}(0)}{dy_2} A'(t', x', z')
\end{aligned}
\tag{1.13}
$$

on the external edge $y_3 \to \infty$ of the viscous sublayer close to the wall which is being considered. Moreover, it follows from the joining conditions that $p_3(t', x', z') = p_2(t', x', z') = p_1(t', x', z')$ and that the latter of these quantities is expressed in terms of the instantaneous displacement thickness $A'(t', x', z')$ by means of equalities (1.8) and (1.10) or by means of (1.9) and (1.10), depending on the Mach number of the approaching stream. As far as the surface $y_3 = 0$ around which the flow occurs is concerned, the sticking conditions $u_3 = v_3 = w_3 = 0$ hold on it.

Let τ_w denote the absolute magnitude of the dimensionless friction applied to the wall in the initial boundary layer. Then,

$$
\frac{dU_{x0}(0)}{dy_2} = \tau_w \tau_x, \quad \frac{dU_{z0}(0)}{dy_2} = \tau_w \tau_z, \quad \tau_x^2 + \tau_z^2 = 1 \tag{1.14}
$$

In order to simplify the final formulation of the boundary value problem, we subject the independent variables together with the required gas parameters to an affine transformation [1–4]:

$$
t' = C^{1/4} \tau_w^{-3/2} T_0 t \tag{1.15}
$$

$$
\begin{gathered}
x' = C^{3/8} \tau_w^{-5/4} T_0^{3/2} x, \quad y_3 = C^{5/8} \tau_w^{-3/4} T_0^{3/2} y, \quad z' = C^{3/8} \tau_w^{-5/4} T^{3/2} z \\
u_3 = C^{1/8} \tau_w^{1/4} T_0^{1/2} u, \quad v_3 = C^{3/8} \tau_w^{3/4} T_0^{1/2} v, \quad w_3 = C^{1/8} \tau_w^{1/4} T_0^{1/2} w \\
p_3 = C^{1/4} \tau_w^{1/2} p, \quad A' = C^{5/8} \tau_w^{-3/4} T_0^{3/2} A
\end{gathered}
$$

As a result of this, the Chapman constant C and the temperature ratio $T_0 = T_w^*/T_\infty^* = R_0^{-1}(0)$ drop out of the differential equations (1.12) and they acquire the canonical form

$$
\begin{gathered}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \frac{\partial p}{\partial y} = 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial y^2}
\end{gathered}
\tag{1.16}
$$

Here, by virtue of (1.8) and (1.10), the pressure is given by

$$
p = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{\partial^2 A(t, \xi, \zeta)/\partial \xi^2}{[(x-\xi)^2 + (1 - M_\infty^2)(z-\zeta)^2]^{1/2}} \, d\zeta \tag{1.17}
$$

in the case of a subsonic boundary layer with $M_\infty < 1$. The expression for the pressure in the case of an approach stream with a supersonic velocity, that is, when $M_\infty > 1$, is found from (1.9) and (1.10):

$$
p = -\frac{1}{\pi} \int_{-\infty}^{x} d\xi \int_{-\delta+z}^{\delta+z} \frac{\partial^2 A(t, \xi, \zeta)/\partial \xi^2}{[(x-\xi)^2 - \beta^2 (z-\zeta)^2]^{1/2}} \, d\zeta \tag{1.18}
$$

$$
\delta = (x - \xi)/\beta,
$$

We note that, in both of the formulas (1.17) and (1.18), the dependence of the Mach number of the particles at infinity is preserved. In the case of spatial perturbations which propagate both in two-dimensional and three-dimensional boundary layers, it is impossible to eliminate this dependence by introducing the difference $|1-M_\infty^2|$ in the similarity transformation (1.15).

It remains to formulate the boundary conditions. When account is taken of (1.14), relationships (1.13), which have to be satisfied on reaching the external edge $y\to\infty$ of the viscous sublayer close to the wall, are transformed to the form

$$u - \tau_x y \to \tau_x A\ (t, x, z), \quad w - \tau_z y \to \tau_z A\ (t, x, z) \tag{1.19}$$

Here, two new parameters, τ_x and τ_z, appear (in fact, it is only their ratio τ_x/τ_z which is important). As far as the conditions for the sticking of the gas particles to the surface $y=0$ around which the flow occurs is concerned, they remain invariant

$$u = v = w = 0 \tag{1.20}$$

The remaining boundary conditions will not be discussed since the subsequent analysis is concerned with free vibrations of infinitely small amplitude which possess a periodic structure with respect to the variables x and z. Furthermore, in order to carry out such an analysis it is convenient to turn to the initial equation (1.6) for the excess pressure while not using the results of its integration, which are expressed by relationships (1.8) and (1.9), which finally lead to (1.17) and (1.18) respectively.

2. THE LINEAR APPROXIMATION

Following the traditional theory of hydrodynamic stability, we will expand the required solution in series in the small parameter Δ, which is proportional to the amplitude of the travelling wave. Confining ourselves to terms which are linear in Δ, we can write

$$(u - \tau_x y,\ v,\ w - \tau_z y,\ p,\ A) = \Delta e^{i(\omega t + kx + mz)}\ [\tau_x f\ (y),\ \ g\ (y),\ \ \tau_z h\ (y),\ \ p_0,\ A_0] \tag{2.1}$$

Substitution of (2.1) reduces (1.16) to a system of ordinary differential equations:

$$\begin{aligned} \frac{dg}{dy} &= -i\,(k\tau_x f + m\tau_z h) \\ \frac{d^2 f}{dy^2} &= i\,(\omega + k\tau_x y + m\tau_z y)\, f + g + \frac{ik}{\tau_x}\, p_0 \\ \frac{d^2 h}{\partial y^2} &= i\,(\omega + k\tau_x y + m\tau_z y)\, h + g + \frac{im}{\tau_z}\, p_0 \end{aligned} \tag{2.2}$$

By virtue of (1.17), the link between the pressure and the displacement thickness at a subsonic velocity of the approach stream will be

$$p_0 = k^2\ [(1 - M_\infty^2)\ k^2 + m^2]^{-1/2} A_0 \tag{2.3}$$

In the case of a supersonic boundary layer with $M_\infty > 1$, it follows from (1.18) that

$$p_0 = \begin{cases} ik^2 \operatorname{sign}(k)\, [(M_\infty^2 - 1)\, k^2 - m^2]^{-1/2} A_0, & m^2 < (M_\infty^2 - 1)\, k^2 \\ k^2\, [m^2 - (M_\infty^2 - 1)\, k^2]^{-1/2} A_0, & m^2 > (M_\infty^2 - 1)\, k^2 \end{cases} \tag{2.4}$$

Both in the case of the positive and negative values which are assumed by the two wave numbers k and m, the square roots of all of the expressions on the right-hand sides of (2.3) and (2.4) remain positive. The quantity m/k specifies the angle at which the wavefront is inclined to the velocity vector of the external unperturbed flow. We denote the Mach angle, which is defined as $\mathrm{tg}\,\alpha_\infty = 1\sqrt{(M_\infty{}^2 - 1)}$, by α_∞. It can be seen from (2.4) that the dependence of the pressure on the displacement thickness in the supersonic boundary layer changes when the angle of inclination of the wave front becomes equal to the Mach angle. This property has far reaching consequences as regards the stability of oblique waves [12].

The limiting conditions

$$f \to A_0, \quad h \to A_0 \quad \text{as} \quad y \to \infty \tag{2.5}$$

for the system of equations (2.2) are obtained from (1.19), while the equalities

$$f = g = h = 0 \quad \text{when } y = 0 \tag{2.6}$$

follow from (1.20). The boundary value problem which has been formulated enables one to calculate the frequency ω of the free vibrations for any specified pair of wave numbers k and m if it is assumed that the Mach number M_∞ and the quantities τ_x and τ_z, which are proportional to components of the surface friction, are fixed.

Let us now consider the reduced wave number $K = k\tau_x + m\tau_x$ and the function $F = k\tau_x f + m\tau_x h$. By differentiating the second and third equations of system (2.2), we obtain

$$d^3F/dy^3 - i\,(\omega + Ky)\,dF/dy = 0 \tag{2.7}$$

Precisely the same equation controls the propagation of Tollmien–Schlichting waves in a Blasius boundary layer with parameters which are constant along the z-axis. The property which has been noted constitutes the essence of a Squire transformation which therefore remains valid when account is taken of the lateral flows in the initial spatial motion of the gas.

The limiting condition

$$F \to KA_0 \quad \text{as} \quad y \to \infty \tag{2.8}$$

for Eq. (2.7) is established using (2.5), and the equalities

$$F = 0, \quad d^2F/dy^2 = i\,(k^2 + m^2)\,p_0, \quad \text{when } y = 0 \tag{2.9}$$

follow from (2.2) and (2.6). The standard technique [13], which rests on the introduction of the new variable $Y = \Omega + i^{1/3}K^{1/3}y$, $\Omega = i^{1/3}\omega K^{2/3}$, $-3\pi/2 < \arg K < \pi/2$ can be used to analyse (2.7). As a result, we arrive at the assertion that its solution, which satisfies the boundary conditions (2.8) and (2.9) exists, if p_0 and A_0 are associated by the relationship

$$\Phi(\Omega) = \frac{i^{1/3}(k^2 + m^2)\,p_0}{K^{5/3}A_0}, \quad \Phi(\Omega) = \frac{d\,\mathrm{Ai}(\Omega)}{dY}\,I^{-1}(\Omega)$$
$$I(\Omega) = \int_\Omega^\infty \mathrm{Ai}(Y)\,dY \tag{2.10}$$

[$\mathrm{Ai}(Y)$ is an Airy function]. A second linear relationship between p_0 and A_0 is established by formulas (2.3) or (2.4). The latter enable one to determine p_0/A_0 at a fixed Mach number M_∞ as a function of the wave numbers k and m. Elimination of p_0/A_0 from (2.3) and (2.10) yields the dispersion relation

$$\Phi(\Omega) = i^{1/3}k^2K^{-5/3}(k^2+m^2)\left[(1-M_\infty^2)k^2+m^2\right]^{-1/2} \tag{2.11}$$

for a subsonic boundary layer with $M_\infty < 1$. In the case of an approaching stream with a supersonic velocity when $M_\infty > 1$, the dispersion relation

$$\Phi(\Omega) = \begin{cases} -i^{4/3}k^2 \operatorname{sign}(k) K^{-5/3}(k^2+m^2)\left[(M_\infty^2-1)k^2-m^2\right]^{-1/2}, & m^2 < (M_\infty^2-1)k^2 \\ i^{1/3}k^2K^{-5/3}(k^2+m^2)\left[m^2-(M_\infty^2-1)k^2\right]^{-1/2}, & m^2 > (M_\infty^2-1)k^2 \end{cases} \tag{2.12}$$

is obtained by eliminating p_0A_0 from (2.4) and (2.10). As (2.12) shows, when the angle of inclination of the travelling wave front reaches the angle at which the intersection of the characteristic surface is inclined to the surface around which the flow occurs a change occurs in the form of the dispersion relationship [12].

3. PROPERTIES OF THE DISPERSION RELATION

In order to establish the symmetry property which is inherent in a dispersion relation, we shall initially assume that the two wave numbers k and m are real and positive (by virtue of the inequalities $-3\pi/2 < \arg K < \pi/2$, negative values of k and m are interpreted as $e^{-i\pi}k$ and $e^{-i\pi}m$, respectively). Let us now assume that the triad of numbers ω, k and m, of which, generally speaking, ω is complex, satisfies the dispersion relation (2.11) or (2.12). Then, the numbers $-\omega_{\text{c.c.}}$, $-k$, $-m$, where the index c.c. denotes a complex conjugate, also represent their solution. In fact, by definition,

$$\Omega(-\omega_{\text{c.c.}}, -k, -m) = -i^{1/3}\omega_{\text{c.c.}}(-k\tau_x - m\tau_z)^{-2/3} = \Omega_{\text{c.c.}} \tag{3.1}$$

By virtue of the properties of the Airy function, we have

$$\Phi(\Omega_{\text{c.c.}}) = \Phi(\Omega)_{\text{c.c.}} \tag{3.2}$$

which denotes a transformation of the left-hand sides of the dispersion relationships to their complex conjugate values. As a result of substituting $-k$ and $-m$ instead of k and m, their right-hand sides are also subjected to an analogous transformation since the square roots of all the expressions occurring in them remain positive.

At fixed k and m, the dispersion relations being considered have a denumerable set of roots, the analysis of which is simplified if one passes from the frequency ω to the argument $\Omega = i^{1/3}\omega K^{-2/3}$ of the function Φ. These roots form an infinite sequence of points in the complex plane Ω. When k and m start to vary, taking only real values, the points are displaced along certain trajectories which constitute a set of dispersion curves. However, a motion of each of these points along its own dispersion curve may take place in both directions even when there is a monotonic increase or decrease in the wave numbers. As the definition (2.10) of the function Φ shows, when $k \to 0$, the points being considered approach points on the real negative semi-axis in an unconstrained manner, the position of these points being given by the zeros of the derivative $d\mathrm{Ai}(\Omega)/dY$. This property can be made use of in order to order the dispersion curves $\Omega_j = \Omega_j(k, m; \tau_x, \tau_z, M_\infty)$ by labelling the roots of the equation $d\mathrm{Ai}(\Omega_{dj})/dY = 0$ with numbers. All of the dispersion curves, with the exception of the first, rest with their other ends at the points $\Omega_{Ij}^{(\pm)}$ which are fixed by the complex conjugate zeros of the integral $I(\Omega)$ which is defined by the last of formulas (2.10). As far as the trajectory of the first root is concerned, it departs to infinity.

We will now describe the results of a more detailed analysis of each of the dispersion relations which takes account of the specific nature of the initial boundary layers in the subsonic and supersonic states. Let us initially put $M_\infty < 1$ and return to (2.11) where, without any loss in the generality, it may be assumed not only that $\tau_x > 0$ but also that $\tau_z > 0$. The dispersion curves for the above-mentioned case are shown in Fig. 1. It is obvious that

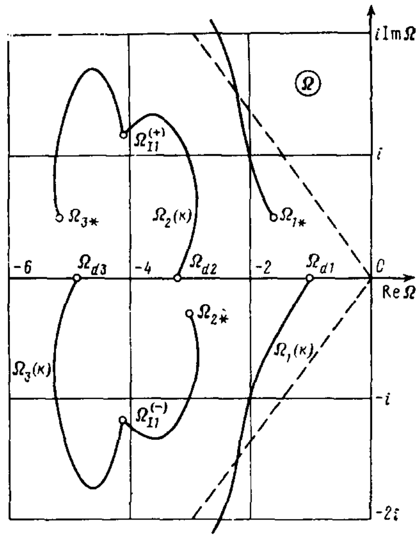

FIG. 1.

they are identical with the curves which are obtained in the analysis of the direct Tollmien–Schlichting waves which propagate in a Blasius boundary layer [13]. However, the motion of an image point along each dispersion curve as k and m are varied possesses features which are only inherent in a spatial flow with $\tau_z \neq 0$. The first of the above-mentioned curves is of the greatest interest since it yields, as will become clear later, the image of the self-perturbing oscillations, while all the remaining curves refer to exponentially decaying pulsations.

We will describe how the passage of the first dispersion curve is completed when $m = m_0 = \text{const}$ and the longitudinal wave number k acquires negative and positive values.

When $k \to -\infty$, we have

$$\Omega_1 \to \infty \exp(5\pi i/6) \tag{3.3}$$

An increase in k leads to downward motion along the curve in Fig. 1 up to a certain limiting point Ω_{1*}, the position of which depends on the sign of the lateral wave number m_0. Initially, let $m_0 > 0$. The above-mentioned point Ω_{1*} is then located in the upper half plane (its coordinates are determined by m_0 and also by τ_x, τ_z and M_∞). On reaching the limiting point, the direction of the motion along the first dispersion curve changes into the opposite direction and, as a result, when $K \to 0$, that is, when $k \to -m_0\tau_z/\tau_x < 0$, we have relationship (3.3) again. The transition through the value $K = 0$ is indicated by the commencement of motion along a branch of the dispersion curve in the lower half plane:

$$\Omega_1 \to \infty \exp(-5\pi i/6) \tag{3.4}$$

if $K \to 0_+$. It is clear that the limiting point of such motion will be the first zero Ω_{d1} of the derivative of the Airy function which is attained when $k = 0$. After rotating at the point Ω_{d1}, the motion along the lower branch of the dispersion curve is completed in the opposite direction up to (3.4) when $k \to \infty$.

The nature of the passage of the first dispersion curve in the case when $m_0 < 0$ is readily established using the symmetry properties, which are expressed by formulas (3.1) and (3.2). A change in the direction of motion along a branch of this curve, which is located in the upper half plane occurs at the point Ω_{d1} when $k = 0$. The following transition through the value $K = 0$ entails the commencement of motion along a branch of the same curve in the lower half plane. The limiting point Ω_{1*} is encountered here and the direction of the motion is reversed. It is important that, when $m_0 \neq 0$, the points Ω_{1*} and Ω_{d1} are separated by a finite distance. If, however, $m_0 = 0$, then $K = k\tau_x$ and (2.11) therefore reduces to the dispersion relationship for direct Tollmien–Schlichting waves in a Blasius boundary layer. The result of this will be a merging of Ω_{1*} and Ω_{d1}

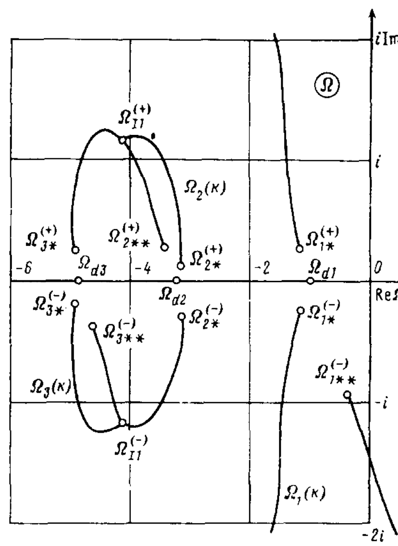

FIG. 2.

which precludes rotation at the double point which has been formed when $k = 0$. It is similar with the passage of the remaining dispersion curves in Fig. 1; the difference reducing to the fact that, when $k \to \pm\infty$, the quantity $\Omega_j \to \Omega_{Ij}^{(\pm)}$.

Let us now consider a supersonic boundary layer, retaining the assumption that $\tau_x > 0$, $\tau_z > 0$ and immediately noting that the value of $\tau_z/\tau_x = 1/\beta$, $\beta = \sqrt{(M_\infty^2 - 1)}$ in a certain sense plays the role of a threshold in the subsequent analysis. When $M_\infty > 1$, the dispersion relationship takes one of two forms (2.12) depending on the relative magnitude of $(m/k)^2$ compared with $M_\infty^2 - 1$. As we did above, let us fix $m = m_0 = \text{const}$. If $k \to -\infty$, $(m_0/k)^2 < M_\infty^2 - 1$ *a fortiori*, whereupon it follows that it is necessary to use the first form of (2.12), in accordance with which we have

$$\Omega_1 \to \infty \exp(-2\pi i/3) \tag{3.5}$$

Next, in the range of values of $k < -|m_0|/\beta$ being considered, let the reduced wave number K be of constant sign, that is, it remains negative. If $m_0 > 0$, then as a consequence of the constraint which has been formulated we shall have the inequality $\tau_z/\tau_x < 1/\beta$, which denotes that the direction of the surface friction lies within the Mach angle α_∞ of the approaching stream. On the other hand, the choice of $m_0 < 0$ does not impose any requirements on the ratio τ_z/τ_x whatsoever. As k increases, an upwards displacement commences along the first dispersion curve in Fig. 2 up to the limiting point $\Omega_{1*}^{(-)}$ which is located in the lower half plane. After rotating at the above-mentioned point, the motion along the dispersion curve is completed in the reverse direction and, when $k \to -|m_0|/\beta$, we get (3.5) again.

The transition to positive values of k which satisfy the previous inequality $(m_0/k)^2 < M_\infty^2 - 1$, produces a commencement of the motion along a branch of the dispersion curve in the upper half plane from

$$\Omega_1 \to \infty \exp(2\pi i/3) \tag{3.6}$$

when $k \to |m_0|/\beta$. Let us assume that the reduced wave number k also does not change sign over the whole range of positive $k > |m_0|/\beta$, that is, it is positive in accordance with its limiting value when $k \to \infty$. If $m_0 < 0$, the inequality $\tau_z/\tau_x < 1/\beta$ follows from the last constraint and only those directions of the surface friction which again fall within the Mach angle α_∞ of the approaching stream satisfy it. On the other hand, for $m_0 > 0$, there are no requirements imposed on the ratio τ_z/τ_x. The downward displacement along the dispersion curve terminates at a certain finite k at the limiting point $\Omega_{1*}^{(+)}$ which lies in the upper half plane. A further increase in k is accompanied by the passage of the dispersion curve in the opposite direction and (3.6) is satisfied when $k \to \infty$. The coordinates of both limiting points $\Omega_{1*}^{(\pm)}$ depend both on m_0 and τ_x, τ_z and M_∞.

Let us now continue the analysis of the first form of the dispersion relationship (2.12) by assuming that there is a change of sign of the reduced wave number K in one of the ranges of k values which have been considered above. A change in sign is only possible subject to the condition $\tau_z/\tau_x > 1/\beta$. This condition holds in the range of negative $k < -m_0/\beta$, if $m_0 > 0$, or it is realized after a transition into the range of positive values $k > |m_0|/\beta$, when $m_0 < 0$. Let the first of these possibilities with $m_0 > 0$ be realized. Motion along the first dispersion curve begins when $k \to -\infty$ from the neighbourhood of the singular point (3.5) in the lower half plane and continues up to the limiting point $\Omega_{1*}^{(-)}$, where $K<0$. However, in the motion along the same curve in the opposite direction, the limit (3.5) is attained due to $K \to 0$ and not by virtue of the fact that $k \to -m_0/\beta$. It can be seen from Fig. 2 that, as a result of passing across the value $K = 0$ an additional special branch of the dispersion curve from

$$\Omega_1 \to \infty \exp(-i\pi/3) \tag{3.7}$$

arises when $K \to 0_-$ on which a new limiting point $\Omega_{1**}^{(-)}$ is located. After rotation at this point, entry to the singular point (3.7) occurs when $k = -m_0/\beta$. The jump to positive $k \to m_0/\beta$ involves the commencement of downward motion along the main branch of the dispersion curve in the upper half plane from the neighbourhood of the singular point (3.6). As is customary, this branch is traversed twice, the change in the directions of the motion occurs at the limiting point $\Omega_{1*}^{(+)}$ and the sign of K is kept positive.

The analysis of the other possibility, which is expressed by the inequality $m_0 < 0$, leads to the following changes in the use of the branches of the first dispersion curve in Fig. 2. These changes are established using the symmetry formulas (3.1) and (3.2). Since the sign of K remains negative for all $k < -|m_0|/\beta$, it is only necessary to return to the main branch of the curve which rests upon the singular point (3.5). Since the change in the sign of K from negative to positive occurs in the interval $k > |m_0|/\beta$, the motion in the upper half plane proceeds both along the main branch and along the auxiliary (singular) branches of the dispersion curve. A singular branch is initially traversed and the jump to an infinitely remote point of this branch

$$\Omega_1 \to \infty \exp(i\pi/3) \tag{3.8}$$

is completed when $k \to |m_0|/\beta$. After a change in sign of K, it is necessary to include the main branch with the singular point (3.6) at infinity in the treatment.

When $m_0 \neq 0$, the limiting points $\Omega_{1*}^{(-)}$ and $\Omega_{1*}^{(+)}$, which are located on the two main branches of the dispersion curve, are separated by a finite distance. If, however, $m_0 = 0$, then $K = k\tau_x$ and the first form of (2.12) therefore reduces to the dispersion relation for direct Tollmien–Schlichting waves which propagate in a supersonic Blasius boundary layer. The result is expressed in the merging of $\Omega_{1*}^{(-)}$ and $\Omega_{1*}^{(+)}$, which represents a rotation at the double point which is formed, this obviously coincides with Ω_{d1}. As far as the two singular branches of the first dispersion curve is concerned, they cease to exist, since a jump in the location of the infinitely remote points (3.7) and (3.8) on them is accomplished either when the sign of K is changed or when $k \to \mp|m_0|/\beta$. In these cases, the equality $m_0 = 0$ implies that $k = 0$. The remaining dispersion curves behave in a completely analogous manner, the difference lying in the fact that, in the limit, $\Omega_j \to \Omega_{lj}^{(\pm)}$ when $k \to \pm\infty$.

It remains to describe the results appertaining to the second form of the dispersion relation (2.12). Since it holds for $(m/k)^2 > M_\infty^2 - 1$ then, at a fixed $m = m_0 = \text{const}$, the limiting value of $k \to \mp\infty$ are excluded from the analysis. If the reduced wave number K changes sign, for which it is necessary to impose the requirement that $\tau_z/\tau_x < 1/\beta$, the nature of the motion along the branches of the first dispersion curve are qualitatively the same as in the case of a subsonic boundary layer which has been analysed above. It is clear from this that, in order to illustrate the basic conclusions, it is necessary once again to turn to Fig. 1 but to take account of the fact that the infinitely remote points depicted in it are reached when the values $k \to \mp|m_0|/\beta$ are taken instead of the earlier limits $k \to \mp\infty$. Retention of the sign of the reduced wave number K over the whole range of variation of k changes the situation: the branch of the main dispersion curve located on it at the point of rotation Ω_{1*} ceases to exist, yielding the origin of the additional (singular) branch shown in Fig. 2. As far as the other branch of the first dispersion curve from the point Ω_{d1} is concerned, motion occurs along it, as usual, in the forward and reverse directions. The total number of branches of the first dispersion curve, which are used in the analysis of a supersonic boundary layer, remains equal to four. Similar remarks hold for the remaining dispersion curves which rest upon their own termini at the points $\Omega_j \to \Omega_{Ij}^{(\pm)}$ when $k \to \mp|m_0|/\beta$. Note that, when $m_0 = 0$, a second form of the dispersion relation is generated by assuming the form $k = 0$.

As can be seen from Figs 1 and 2, the dispersion curves, which are images of the trajectories of the roots of the dispersion relation (2.11) and (2.12), do not intersect for arbitrary real k and m. It follows from this that any two functions $\omega_p = \omega_p(k, m; \tau_x, \tau_z, M_\infty)$ and $\omega_q = \omega_q(k, m; \tau_x, \tau_z, M_\infty)$ determined by them take different values for the same k and m if $p \neq q$. In other words, there are no branching points in the complex plane ω if k and m are confined to being real numbers.

4. STABILITY ANALYSIS

The decay or build-up of oscillations is determined by arg ω of the complex frequency ω and, in fact, the amplitude of the pulsations increases exponentially subject to the condition $-\pi < \arg\omega < 0$. Since $\arg K$ can only be equal to 0 or to $-\pi$ when the two wave numbers k and m are chosen to be real both the subsonic and supersonic boundary layers lose stability in the intervals of change in $\arg\Omega$ which are specified by the following inequalities:

$$\begin{aligned} -5\pi/6 < \arg\Omega < \pi/6, &\quad \text{for } \arg K = 0 \\ -\pi/6 < \arg\Omega < 5\pi/6 &\quad \text{for } \arg K = -\pi \end{aligned} \tag{4.1}$$

We will now consider subsonic flow. Rays, with a slope which is fixed by $\arg\Omega = 5\pi/6$ and $\arg\Omega = -5\pi/6$, are represented by the dashed lines in Fig. 1. These rays only intersect the first dispersion curve with the same entry angles at the infinitely remote singular points $\Omega_1 \to \infty \exp(\pm 5\pi i/6)$, that is, a range of real values of k and m exist in the case of the named curve which ensure that the inequalities (4.1) are satisfied. It is concluded from this that the first root of the dispersion relation (2.11) can represent the parameters (frequency and pair of wave numbers) of unstable free vibrations. As far as all the remaining roots are concerned, they correspond to perturbations with an amplitude which decays exponentially with time. Hence, the analysis of the stability of a spatial boundary layer with $M_\infty > 1$ repeats to a significant extent that which was developed in [13] as applied to direct Tollmien–Schlichting waves which propagate in two-dimensional flow in the case of a plate (the difference reduces to the traversal of the dispersion curves in the forward and reverse directions).

The question of the stability of a supersonic spatial boundary layer is solved somewhat more subtly. We begin with the second form of the dispersion relationship (2.12) which holds subject to the condition that $(m/k)^2 > M_\infty^2 - 1$. Since this form becomes degenerate at $k = 0$ when $m = 0$, its application is exclusively associated with oblique Tollmien–Schlichting waves. The existence of a range over which they are unstable is readily established using the same Fig. 1. Taking account of the relationship $k/m = \operatorname{tg}\gamma$, where we denote by γ the angle of inclination of the wave front to the velocity vector of the external unperturbed flow, we emphasize that the oblique waves under consideration propagate in directions which satisfy the condition $\gamma < \alpha_\infty$. In the case of such waves, the projection of the velocity of the external flow on the normal to the front is smaller than the local velocity of sound, that is, the mechanism of the instability of a supersonic spatial boundary layer is essentially explained in the given case by factors which cause an exponential increase in the oscillations in a viscous flow at subsonic velocities.

By putting $(m/k)^2 < M_\infty^2 - 1$, we return to the first form of the dispersion relationship (2.12). All the oblique Tollmien–Schlichting waves which propagate in a two-dimensional Blasius boundary

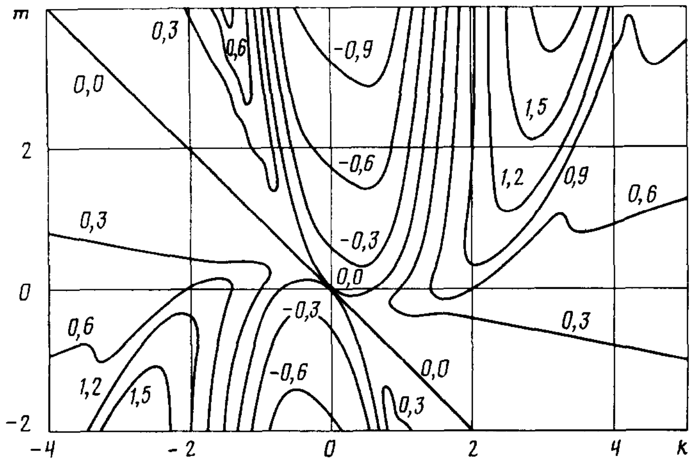

FIG. 3.

layer subject to such a constraint are stable [12]. Actually, the values of Ω calculated using their parameters form the main branches of the first dispersion curve in Fig. 2 with the singular points $\Omega_1 \to \infty \exp(\mp 2\pi i/3)$ which do not fall within either of the intervals (4.1). The presence of a lateral component τ_z in the surface tension leads to a new instability mechanism. It is clear that self-exciting oscillations can only be represented by special branches of the dispersion curve which are fixed by the infinitely remote points $\Omega_1 \to \infty \exp(\mp\pi i/3)$ within (4.1): the form of the main branches is independent of τ_z (only the positions of the points of rotation $\Omega_{1*}^{(+)}$ on them changes). However, as was established above, the formation of special branches is caused by a change in sign of the reduced wave number K, for which it is required that the condition $\tau_z/\tau_x = 1/\beta = \alpha_\infty$ should be satisfied. Hence, a supersonic spatial boundary layer can lose stability if the direction of the surface friction in the plate reaches beyond the Mach angle of the approach stream. As far as the angle of inclination of the wave front to the velocity vector of the external stream, γ, is concerned, this angle $\gamma > \alpha_\infty$ also.

For a more complete analysis of the properties of unstable motion under subsonic and supersonic conditions let us consider the level lines of the function $\operatorname{Im}\omega_1$ by assigning real values to the wave numbers k and m. The first of these cases is illustrated in Fig. 3, in the construction of which we put $M_\infty = 0.1$, $\tau_x = \tau_z = \sqrt{2}/2$. In particular, we note that the shape of the curves in the k, m plane confirms the symmetry property of the roots of the dispersion relationship found above, which is expressed by means of formulas (3.1) and (3.2)

We will prove the fundamentally important fact that domains exist where $\operatorname{Im}\omega_1$ is not only smaller than zero but also increases in absolute magnitude as $|k|$ and $|m|$ increase. In other words, the Cauchy problem for the linearized system of equations (1.16) with the pressure introduced using (1.17) is, generally speaking, ill-posed according to Petrovskii [14].

For this purpose, let us consider the first quadrant of the k, m plane: it is seen from Fig. 3 that the third quadrant possesses an analogous structure. Let us put $m = ck_3$, $c > 0$ and let $k \to \infty$. To a first approximation, the dispersion relationship (2.11) then reduces to the form

$$\Phi(\Omega) = i^{1/3} c^{-2/3} \tau_z^{-5/3}, \quad \omega = i^{-1/3} \Omega c^{2/3} \tau_z^{2/3} k^2 \tag{4.2}$$

on the right-hand side of which neither k nor m appear. Let us compare (4.2) with the dispersion relationship for direct Tollmien–Schlichting waves which propagate in an incompressible Blasius boundary layer which corresponds to $m = \tau_z = 0$. In the latter case, the amplitude of the perturbations remains constant in time if the wave number $k = k_* = 1.0005$. It follows from this that the constant $c = c_* = k_*^{-2}\tau_z^{-5/2} = 0.999\tau_z^{-5/2}$ specifies neutral oscillations in a subsonic spatial boundary layer for any $M_\infty < 1$.

The curve $m = c_* k^3$ serves as an asymptote which separates the self-exciting pulsations for which $c < c_*$ from the stable waves with $c > c_*$. Furthermore, it is possible to separate out from among the perturbations in an incompressible Blasius boundary layer those perturbations whose amplitude growth increments attain external values. In the terms adopted in [13], the corresponding wave numbers are defined as $k = k^* = 2.716$ and $k = k_4^* = 4.346$ for the maxima, while $k = k_3^* = 3.616$, which realizes a minimum, lies between them. It is clear that $c = c_2^* = (k_2^*)^{-2}\tau_z^{-5/2} = 0.1356\tau_z^{-5/2}$; $c = c_3^* = (k_3^*)^{-2}\tau_z^{-5/2} = 0.07648\tau_z^{-5/2}$ and $c = c_* = c_4^* = (k_4^*)^{-2}\tau_z^{-5/2}$ also possess similar extremal properties: on substituting them into the right-hand side of (4.2), they yield solutions Ω_1 with local negative minima $\operatorname{Im}(i^{-1/3}\Omega_{12}{}^*c_2{}^{*2/3}\tau_z{}^{2/3})$ and $\operatorname{Im}(i^{-1/3}\Omega_{14}{}^*c_4{}^{*2/3}\tau_z{}^{2/3})$ and the local negative maximum $\operatorname{Im}(i^{-1/3}\Omega_{13}{}^*c_3{}^{*2/3}\tau_z{}^{2/3})$. It follows from this that the rate of growth of the amplitude of the self-exciting pulsations being considered, for which the constant $c < c_*$, obeys the estimate $\operatorname{Im}\omega_1 \sim -k^2 \to -\infty$ in the limit as $k \to \infty$. The assertion concerning the ill-posed nature of the Cauchy problem is therefore proved.

The contours of the domain around the first principal maximum $\operatorname{Im}(i^{-1/3}\Omega_{12}{}^*c_2{}^{*2/3}\tau_z{}^{2/3})$ can be clearly seen in Fig. 3. The irregularities in the behaviour of the level line of $\operatorname{Im}\omega_1$ are associated with the local maximum $\operatorname{Im}(i^{-1/3}\Omega_{13}{}^*c_3{}^{*2/3}\tau_z{}^{2/3})$ and, secondly, with the more weakly expressed minimum $\operatorname{Im}(i^{-1/3}\Omega_{14}{}^*c_4{}^{*2/3}\tau_z{}^{2/3})$.

We may arrive at the conclusion that the Cauchy problem is ill-posed for the linearized system of equations (1.16) and (1.17) by also considering the second quadrant of the k, m plane shown in Fig. 3. The fourth quadrant has a similar structure. Putting $m = ck^3$, $c < 0$, we shall assume that $k \to -\infty$. It follows from the fact that $m > 0$ that $K > 0$ and, hence, both relationships (4.2) remain true if c is replaced in them by $|c|$. It follows from what has been said above that the curve $m = c_* k^3$ with the coefficient $c_* = 0.999\tau_z^{-5/2}$ is the second branch of the asymptote which separates the self-exciting pulsations from the stable waves with an exponentially decaying amplitude. A domain is clearly separated out to the left of this branch whose existence is attributable to the principal minimum $\operatorname{Im}(i^{-1/3}\Omega_{12}{}^*c_2{}^{*2/3}\tau_z{}^{2/3})$. It is difficult to distinguish the irregularities in the lines of the level of $\operatorname{Im}\omega_1$ which are generated by the local maximum $\operatorname{Im}(i^{-1/3}\Omega_{13}{}^*c_3{}^{*2/3}\tau_z{}^{2/3})$ and the second minimum $\operatorname{Im}(i^{-1/3}\Omega_{14}{}^*c_4{}^{*2/3}\tau_z{}^{2/3})$. It is fundamentally important that, in the limit as $k \to -\infty$, the rate of growth of the amplitude of the self-exciting pulsations with a constant $|c| < c_*$ should satisfy the estimate $\operatorname{Im}\omega_1 \sim -k^2 \to -\infty$.

We will now examine the results of an analysis of oblique Tollmien–Schlichting waves which propagate in a supersonic spatial boundary layer. Again, let $m = ck^3$, where the constant can be both positive as well as negative and let us put $k \to \pm\infty$, respectively. Such perturbations obey the second of the forms of the dispersion relation (2.12) which, to a first approximation, reduces to (4.2) with c replaced by $|c|$. The conclusion concerning the Petrovskii ill-posed nature of the Cauchy problem for the linearized system of equations (1.16) with the pressure defined by relation (1.18) immediately follows from this. In this case, the ill-posed nature of the problem is caused by waves with a direction of the normal to the front which exceeds the Mach angle α_∞ of the external flow. Furthermore, the vortices in such waves (as in the analogous waves moving in a subsonic boundary layer) are elongated in a direction which is very close to the direction of the main flow.

In concluding we note a feature in the solution of the dispersion equations which results from the passage of the reduced wave number K through zero. The obvious sense of the equality $K = k\tau_x + m\tau_z$ lies in the requirement that the wave vector should be orthogonal to the vector of the surface friction applied to the wall. Using (2.1) in the case of a subsonic boundary layer in the limit when $K\to 0$ ($k\to k_0$, $m\to m_0=-k_0\tau_z/\tau_x$), we have

$$\omega_1=-\frac{c_0^2}{K}-\frac{\sqrt{2}}{2}(1+i)\frac{K^3}{c_0}+\ldots$$

$$c_0^2=\frac{k_0^2(k_0^2+m_0^2)}{[(1-M_\infty^2)k_0^2+m_0^2]^{1/2}} \tag{4.3}$$

A similar feature has recently been pointed out in the case of Gertler vortices in an incompressible fluid which flows around a warped cylindrical surface with genetratrices which are perpendicular to the velocity vector of the basic motion [15–17]. Although $\operatorname{Im}\omega_1=\sqrt{2}K^3/(2c_0)\to 0$ here together with $K\to 0$, the exceedingly pronounced singularity in $\operatorname{Re}\omega_1$ may turn out to have a substantial effect on the propagation of complex wave structures (of wave packets, for example).

In the case of free vibrations with $m^2>(M_\infty^2-1)k^2$ which occur in a supersonic boundary layer, the first of formulas (4.3) remains true if by the constant c_0 one understands

$$c_0^2=\frac{k_0^2(k_0^2+m_0^2)}{[m_0^2-(M_\infty^2-1)k_0^2]^{1/2}}$$

In this case, as was shown in the preceding section, $\tau_z/\tau_x<1/\beta$, that is the direction of the surface friction falls within the Mach angle α_∞ of the approach stream.

On the other hand, let the velocity field in the initial spatial boundary layer be such that $\tau_z/\tau_x>1/\beta$. As an example, the lines of the level of the function $\operatorname{Im}\omega_1$ in the plane of real wave numbers k and m are plotted in Fig. 4. The parameters of the unperturbed motion were chosen as follows: $M_\infty=2.1$, $\tau_x=\sqrt{3}/2$ and $\tau_z=1/2$. The surface friction vector is now very strongly inclined from the direction of the external flow and lies outside the Mach angle α_∞ corresponding to it. It was established in the preceding section that the transition through the value $K=0$ in a boundary layer with the structure being considered can only be attained in waves which satisfy the condition $m^2<(M_\infty^2-1)k^2$ which means that the angles of inclination of their fronts with respect to the direction of the external flow exceeds the Mach angle α_∞. Returning to the first of the forms of the dispersion relationship (2.12), we find

$$\omega_1=i\,\operatorname{sign}(k)\frac{c_0^2}{K}+\ldots,\quad c_0^2=\frac{k_0^2(k_0^2+m_0^2)}{[(M_\infty^2-1)k_0^2-m_0^2]^{1/2}}$$

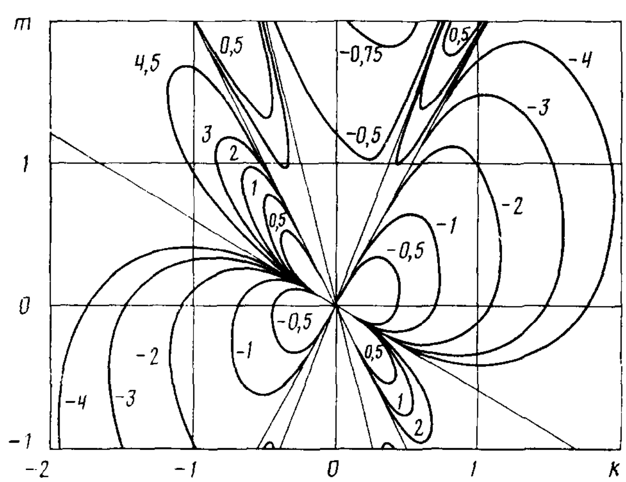

FIG. 4.

It is obvious from this that the transition through $K = 0$ implies a change in sign of $\operatorname{Im}\omega_1$ where here $\operatorname{Im}\omega_1 = -c_0^2/K \to -\infty$ when $k<0$ and $K \to 0_+$ and, similarly $\operatorname{Im}\omega_1 = -c_0^2/|K| \to -\infty$ when $k>0$ but $K \to 0_-$. Hence, even in the case of finite values of the two wave numbers k and m, the Cauchy problem for the linearized equations (1.16) and (1.18) is ill-posed according to Petrovskii if the vector of the viscous friction on the bottom of the boundary layer deviates sufficiently strongly from the direction of the supersonic flow on its external edge. Actually, the reason why the problem is ill-posed is the long-wavelength vortices which are extended along the direction of the surface friction (as a rule, the pulsations from the short-wavelength part of the spectrum possess this property). In the final analysis, the spatial inhomogeneities which were initially in the initial boundary layer or which arose during the initial phase of the development of unstable oscillations are capable of exerting a powerful destabilizing effect on the whole course of this process.

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